E1: Midterm Exam *Solutions*

CEE 260/MIE 273: Probability & Statistics in Civil Engineering

October 10, 2019

time limit: Seventy Minutes

Name

Please print your name clearly in the box below.

Turn to the next page to read the instructions.

Instructions

This exam contains **16 pages** (including the front and back pages) and **7 problems**. You have **70 minutes** to complete it. Check to see if any pages are missing. Make sure you have written your name on the front page. If for any reason you have any loose pages, put your initials on the top of these pages.

You may use *only* **writing materials** and a **calculator** on this exam, along with the **formula sheet** and **probability tables** provided. However, you may NOT consult or use *past year exams, lecture notes, slides, homework, phones, computers, tablets or other electronic device*. If you have any of the disallowed items, please bring them to the front of the lecture room before you begin your exam.

The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Show ALL your work where appropriate**. The work you show will be evaluated as well as your final answer. Thus, provide ample justification for each step you take. Indicate when you have used a probability table to obtain a result. In the long response questions, simply putting down an answer without showing your steps will not merit full credit. **EXCEPTION:** For short response or "True/False" questions, *no explanations are required*. However, the more work you show, the greater your chance of receiving partial credit if your final answer is incorrect.
- If you need more space, use the blank pages at the end, and clearly indicate when and where you have done this.
- Questions are roughly in order of the lectures, so later questions may not necessarily be harder. If you are stuck on a problem, it may be better to skip it and get to it later.
- Manage your time wisely. Do not spend too much time on problems with fewer points.

Do not write anything on this page. Please turn over.

Problem	**Score**	**Points**
1		**6**
2		**10**
3		**5**
4		**5**
5		**7**
6		**7**
7		**10**
TOTAL		**50**

Problem 1 *True/False questions (6 points)*

Respond *True* ("T") or *False* ("F") to the following statements.

[1pt] **(i)** $P(A \cup B)$ is always equal to $P(A) + P(B)$. **F**

[1pt] **(ii)** If two events A and B are not mutually exclusive, then $P(AB) > 0$. **T**

[1pt] **(iii)** If two random variables A and B are statistically independent, then $Cov(A, B) > 0$. **F**

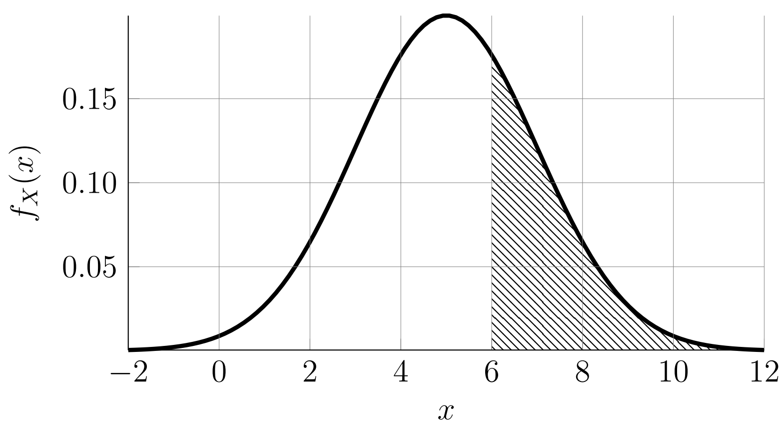
[1pt] **(iv)** The CDF of a random variable X is defined as $P(X \leq x)$. **T**

[1pt] **(v)** If the natural logarithm of a random variable X is normally distributed, then $E(\ln X) = \mu_X$. **F**

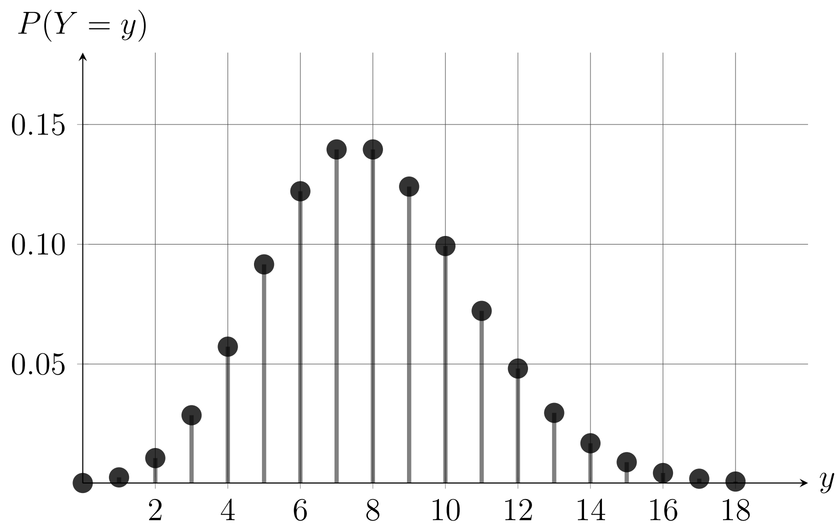
[1pt] **(vi)** $P(A|B)P(B) - P(B|A)P(A) = 1$. **F**

Problem 2 *Short answer questions (10 points)*

[2pts] **(i)** Shade the area under the curve that gives you the probability $P(X \geq 6)$.

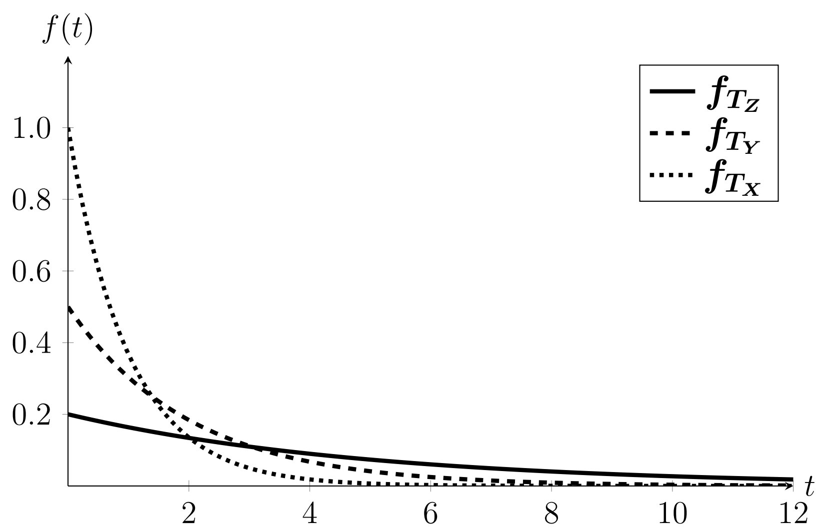


(ii) The PMF of a random variable Y is given in the figure below. Use the figure to estimate the probability $P(Y = 10 \cup Y = 12)$. *[2pts]*



Answer: $P(Y = 10 \cup Y = 12) \approx 0.10 + 0.05 = 0.15.$

(iii) Consider the PDFs of the epxonential random variates T_X, T_Y and T_Z shown in the figure below. Which of them has the greatest mean? *[1pt]*



Answer: $\boldsymbol{T_Z}$ (since $E(T_Z) = \frac{1}{0.2} = 5 \geq E(T_Y) = 2 > E(T_X) = 1$)

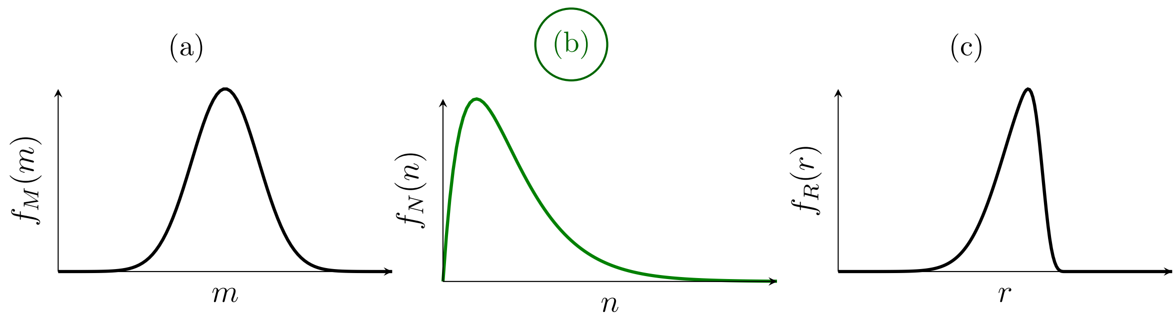
[2pts] **(iv)** Given that the parameter λ of a lognormal random variate is 5.2. What is the median of this distribution?

$$\lambda = \ln x_m = 5.2$$

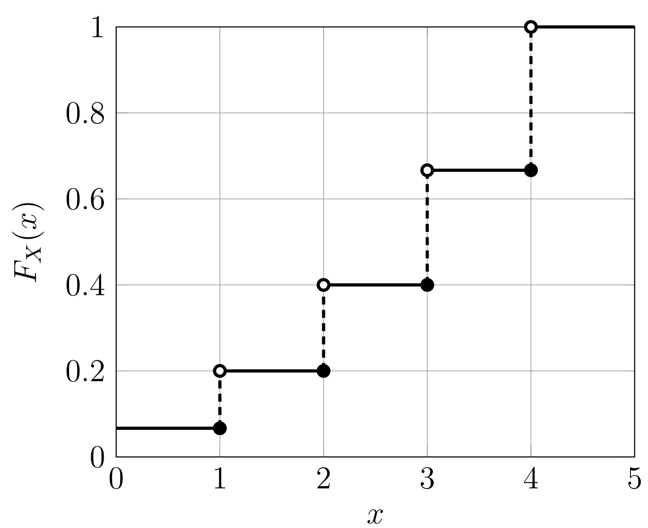
$$\implies x_m = e^{\lambda} = e^{5.2} = 181.27$$

The median is $\boxed{181.27}$. *(1 pt for correct formula or substitution; 1 pt for final answer)*

[1pt] **(v)** Which of the distributions below has a positive skewness? Circle (a), (b) or (c).



[2pts] **(vi)** The figure below shows the CDF of a discrete random variable X. To aid you in properly interpreting this plot, you are given that $P(X \leq 2) = 0.2$. What is the probability $P(X > 3)$?



Answer: $P(X > 3) = 1 - 0.4 = 0.6$ *(1pt: relevant work, answer wrong)*

Problem 3 *Events and probability (5 points)*

The mass transit sector in Massachusetts is subject to labor strikes by the drivers (event D), mechanics (event M) and ticket attendants (event T). The probability of strikes by each of the individual groups in the next year is given by:

$$P(D) = 0.08$$
$$P(M) = 0.05$$
$$P(T) = 0.04$$

Assuming that strikes by the individual groups are statistically independent of each other, find the probability of a labor strike in the mass transit sector in Massachusetts next year.

$$\begin{aligned}
P(D \cup M \cup T) &= 1 - P(\overline{D \cup M \cup T}) \\
&= 1 - P(\overline{D}\,\overline{M}\,\overline{T}) \\
&= 1 - P(\overline{D})P(\overline{M})P(\overline{T}) \\
&= 1 - (1 - 0.08)(1 - 0.05)(1 - 0.04) \\
&= 1 - (.92)(.95)(.96) \\
&= \boxed{0.161}
\end{aligned}$$

Rubric: 1 point for each of the following:

- Properly formulating probability
- Applying complement rule
- Applying de Morgan's rule
- Applying statistical independence
- Computing answer correctly
- IF addition rule for 3 events is correctly implemented, award full points.

Problem 4 *Binomial distribution (5 points)*

The probability of winning a construction job is 0.6. If a contractor submits bids to 8 construction jobs, what is the probability the contractor will win at least 3 jobs? (Assume that winning each job is an independent event.)

$$
\begin{aligned}
\textit{[1pt]}\ P(X \geq 3) &= 1 - P(X \leq 2) && \textit{[1pt]} \\
&= 1 - \sum_{k=0}^{2} \binom{8}{k} (0.6)^k (0.4)^{n-k} && \textit{[2pts]} \\
&= 1 - 0.0498 \quad \text{(from table)} \\
&= \boxed{0.9502} && \textit{[1pt]}
\end{aligned}
$$

Rubric:

- 2 points for correctly expressing probability
  - For instance: writing $P(X \geq 3)$ attracts 1 point
  - If this is not correctly translated $1 - P(X \leq 2)$ then the second point is forfeited.
- 2 points for writing out correct binomial CDF
  - However, if ONLY $P(X \geq 3)$ is written, followed by the upper part of the CDF, i.e.
    $$P(X \geq 3) = \sum_{k=3}^{8} \binom{8}{k} (0.6)^2 (0.4)^6$$
    then, 4 points are awarded in total for this.
  - Due to the error in the Formula Sheet, if the upper limit of the sum is written as 8 instead of 2, please award full points for CDF and correct the error.
- 1 point for correctly computing final asnwer

Problem 5 *Exponential distribution (7 points)*

The delay time of a flight is exponentially distributed with $v = 3$ (mean rate of occurrence per hour).

(a) Show that the probability that a flight is delayed by at most 15 minutes (a quarter of an hour) is approximately 0.528. *[3 pts]*

$$P(T \leq 0.25) = 1 - e^{-3(0.25)}$$
$$= \boxed{0.528}$$

Rubric:

- 1 point for correctly expressing probability
- 1 point for correctly substituting parameters into CDF
- 1 point for computing final answer correctly

(b) Given that a family member has already waited for 30 minutes, what is the probability that a certain flight will be further delayed by over 30 minutes when it arrives? (In other words, find $P(T > 1|T > 0.5)$). *[4pts]*

Use the memoryless property:

$$P(T > 1|T > 0.5) = P(T > 0.5)$$
$$= e^{-3(0.5)}$$
$$\approx \boxed{0.223}$$

Rubric: 1 point for each of the following

- Writing conditional probability or invoking memo
- Simplifying conditional probablility either fully or by invoking memoryless property
- Correctly substituting parameters
- Computing final answer correctly

Problem 6 *Joint distributions (7 points)*

The joint PMF of two random variables X and Y is shown in the table below.

	$p_{X,Y}(x,y)$	y = **0**	**1**	**2**
	0	0.10	0.04	0.02
x	**1**	0.08	0.20	0.06
	2	0.06	0.14	0.30

[1pt] **(i)** Find $P(X = 1, Y = 2)$.

$$P(X = 1, Y = 2) = \boxed{0.06}$$

[2pts] **(ii)** Compute $P(X \geq 1)$.

$$\begin{aligned} P(X \geq 1) &= 1 - (0.10 + 0.04 + 0.02) \\ &= 1 - 0.16 \\ &= \boxed{0.84} \end{aligned}$$

Alternatively, one could sum all the probabilities in the second and third rows of the matrix to obtain the answer.

[2pts] **(iii)** Compute $P(X \geq 1|Y = 2)$.

$$\begin{aligned} P(X \geq 1|Y = 2) &= \frac{P(X \geq 1 \cup Y = 2)}{P(Y = 2)} \\ &= \frac{0.06 + 0.30}{0.02 + 0.06 + 0.30} \\ &= \frac{0.36}{0.38} \approx \boxed{0.9474} \end{aligned}$$

Rubric: 1 point if denominator not considered and answer given as 0.36.

[2pts] **(iv)** Are X and Y statistically independent? Justify your answer.

No they are not. *[1pt]* This is because $P(X \geq 1) \neq P(X \geq 1|Y = 2)$. *[1pt]*

Other correct reasons are acceptable.

Problem 7 *Functions of multiple random variables (10 points)*

The differential settlement between two piles in a foundation is given by:

$$D = S_1 - S_2 \tag{1}$$

where S_1 and S_2 are normally distributed.

You are given the following:

$$\mu_{S_1} = \mu_{S_2} = 3.0 \tag{2}$$

$$\sigma_{S_1} = \sigma_{S_2} = 0.7 \tag{3}$$

$$\rho = 0.8 \tag{4}$$

First, convince yourself that the mean of D is given by

$$\mu_D = \mu_{S_1} - \mu_{S_2} = 3 - 3 = 0 \tag{5}$$

(i) Now, show that the variance of D is equal to 0.196. *[4pts]*

$$\begin{aligned}
Var(D) &= Var(S_1) + Var(S_2) - 2Cov(S_1, S_2) \\
Cov(S_1, S_2) &= \rho\sigma_{S_1}\sigma_{S_2} \\
&= 0.8(0.7)(0.7) = 0.392 \\
\therefore Var(D) &= \sigma_{S_1}^2 + \sigma_{S_2}^2 - 2(0.392) \\
&= 0.7^2 + 0.7^2 - 0.784 \\
&= 0.98 - 0.784 \\
&= \boxed{0.196}
\end{aligned}$$

Rubric:

- Students may use incorrect relationships betweeen variance and standard deviation. Deduct 1 point if wrong relationship used, but variance computation still consistent with formula written down.
- Award 1 point for correct computation of covariance
- 1 point for explicitly showing the terms in the variance formula give 0.196

[6pts] **(ii)** Find the probability that the magnitude of the differential settlement $|D|$ is no more than 0.5 inch.

$$\begin{aligned}
P(|D| \leq 0.5) &= P(-0.5 < D \leq 0.5) \\
&= \Phi\left(\frac{0.5 - 0}{\sqrt{0.196}}\right) - \Phi\left(\frac{-0.5 - 0}{\sqrt{0.196}}\right) \\
&= \Phi(1.129) - \Phi(-1.129) \\
&= \Phi(1.129) - (1 - \Phi(1.129)) \\
&= 0.871 - 1 + 0.871 \\
&= \boxed{0.741}
\end{aligned}$$

Rubric:

- 1 point writing $P(D \leq 0.5)$. IF the correct probability $P(|D| \leq 0.5)$ is written, award 2 points instead.
- If both parts of the probability are correctly computed, then award 4 points.
- 1 point can be taken off for the following:
  - Using wrong mean or standard deviation
  - Obtaining wrong probability from table
  - Only computing the probability $P(D \leq 0.5)$.
- Thus, if the final answer is 0.871 and all work is properly shown, 4 points should be awarded overall
- If the final answer is correctly given as 0.741 and all work is properly shown, then 6 full points should be awarded.

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CEE 260/MIE 273 | J. B. Oke | Fall 2019

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