CEE 260/MIE 273: Probability and Statistics in Civil Engineering
Lecture M4c: Hypothesis Testing and p-values

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Outline

Today's objectives

- Learn how to conduct a hypothesis test on the mean of a population
- Know when to use a lower-tailed, upper-tailed or two-tailed test
- Understand Type I errors and their relationship to *p*-values
- Learn how to use *p*-values to conduct a hypothesis test

Hypothesis testing

- Hypothesis testing provides a framework for evaluating parameter(s) of a population with respect to a desired or known outcome.
- Given that the in most cases, we can only estimate these parameters, hypothesis testing allows us to determine if the estimate supports a **research hypothesis**.
- The results of this testing is useful for **decision-making**.

Formulating a hypothesis test

A hypothesis is a statement regarding a parameter.
In a test, there are usually two competing hypotheses:

- H_0: the **null** hypothesis
- H_1: the **alternative** hypothesis (H_A is also used to denote this)

The null hypothesis is usually framed as an equality, i.e.:

$$H_0 : p = p_0 \tag{1}$$

where p_0 is the specified standard.
The alternative is given by

$$H_1 : p \neq p_0 \tag{2}$$

Outcomes of a hypothesis test

The null hypothesis is presumed unless there is sufficient evidence to discard it. The alternative hypothesis, however, is what we hope to support.

> *In experimental design, we frame the null hypothesis in such a way as to reject it.*

Thus there are **two outcomes** of a hypothesis test:

- **Reject H_0**: because of sufficient sample evidence in support of H_1
- **Fail to reject H_0**: because of insufficient evidence in support of H_1

No truth test for the null hypothesis

The failure to reject H_0 does not mean that H_0 is true.

Further explanation of hypothesis test outcomes

Example 1: Outcome of a trial

In a jury trial, the hypotheses are:

- H_0: defendant is innocent
- H_1: defendant is guilty (not innocent)

The null hypothesis H_0 is **rejected** if there evidence beyond reasonable doubt that the defendant is guilty.

However, **failure to reject** H_0 does not imply the defendant is innocent, only that there is **insufficient evidence to prove otherwise**.

Hypothesis testing in practice

Example 2: Chip manufacturing

A company manufacturing RAM chips claims the defective rate of the population is 5%. Using a 500-chip sample from production, formulate a hypothesis test to evaluate the validity of the company's claim.
Let p denote the *true* defective probability.
We structure our hypothesis test as follows:

$$H_0 : p = 0.05$$
$$H_1 : p > 0.05$$

Note: this is a one-sided hypothesis test (testing in one direction only)

Hypothesis testing in practice

Example 2: Chip manufacturing (cont.)

In order to test the hypotheses, we must choose a **test statistic**.
Here, we let X denote the number of defective chips in the sample of 1000.
Then in order to determine whether or not to reject H_0, we must decide on a **critical value**.
We note that this is a Bernoulli process. Thus, if $p = 0.05$, then the expected number of defective chips is

$$\overline{X} = np = 1000 \times 0.05 = 50$$

Say the critical value were $p^* = .1$, then $p \geq .1$ could then be considered as strong evidence that $p > 0.05$.

Hypothesis testing in practice

Example 2: Chip manufacturing (cont.)

Thus, we would reject H_0.

Errors in hypothesis testing

Since we are working with finite samples, errors are bound to occur in decision-making.

The decision matrix is:

	H_0 is true	H_1 is true
Fail to reject H_0	Correct decision	Type II error
Reject H_0	Type I error	Correct decision

Type I error

Definitions

- The incorrect rejection of H_0 is a Type I error.
- Also known as a *false positive*
- The probability of a Type I error is the **level of significance**, α

Examples of Type I error

- Convicting a defendant of a crime when they are innocent (Example 1)
- Diagnosing a patient with a disease when in fact they do not have it (i.e. the null hypothesis is that the disease is NOT present)

Level of significance

A Type I error is less likely as α reduces.
We revisit Example 2.

Example 2: Chip manufacturing (cont.)

Find the level of significance α when the critical value of p is .1.
In other words, what is the probability of incorrectly rejecting H_0 when it is true?

$$
\begin{aligned}
\alpha &= P(\text{Type I error}) && (3)\\
&= P(p > 0.1) && (4)\\
&= 1 - \Phi\left(\frac{p^* - p_0}{SE_{p_0}}\right) = 1 - \Phi\left(\frac{.1 - .05}{\sqrt{.05(.95)/1000}}\right) && (5)\\
&= \boxed{0} && (6)
\end{aligned}
$$

Type II errors

Definitions

- Failure to reject H_0 when in fact H_1 is true is a Type II error.
- Also known as a *false negative*
- The probability of a Type II error is denoted $\boldsymbol{\beta}$

Note

We cannot compute β except the alternative hypothesis H_1 is specified.
Much of our focus will be on dealing with the level of significance, α.

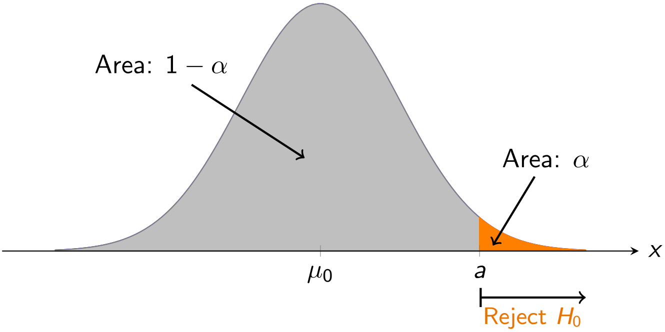
Summary of hypothesis testing approach

1. *Define* the **null** (H_0) and **alternative** (H_1) hypotheses
2. *Determine* the appropriate **test statistic** (and distribution)
3. *Estimate* the test statistic from the sample data
4. *Specify* or *identify* the **level of significance** (α)
5. *Define* the **region of rejection/critical region** of the null hypothesis by choosing the **critical value**.
6. *Decide.* If the test statistic is in the critical region, reject H_0. If not, do not reject H_0 (fail to reject it)

One-sided tests

Case A: upper tail

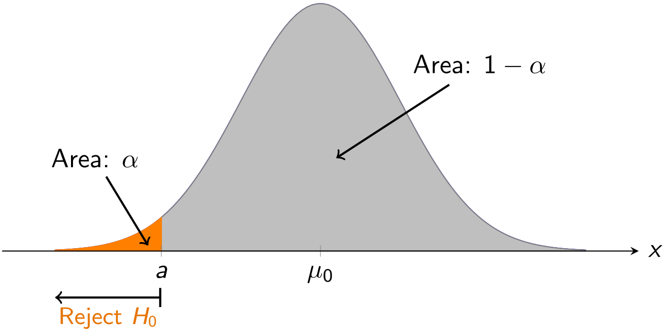
- $H_0 : \mu = \mu_0$
- $H_1 : \mu > \mu_0$



One-sided tests (cont.)

Case B: lower tail

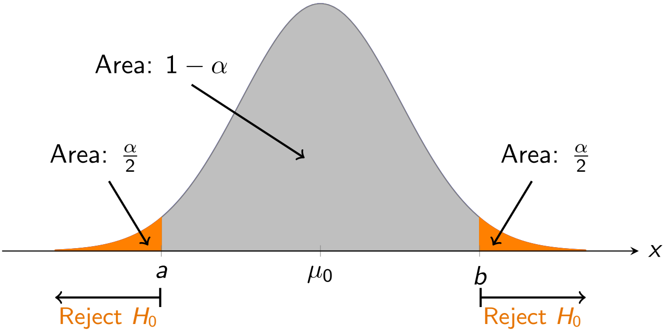
- $H_0 : \mu = \mu_0$
- $H_1 : \mu < \mu_0$



Two-sided tests

Case C: both tails

- $H_0 : \mu = \mu_0$
- $H_1 : \mu \neq \mu_0$



Distribution of the test statistic

In this lecture, the test statistic is the **sample proportion**.
We will assume the normal distribution is the success-failure condition holds.

The sample proportion is **normally** distributed and its variance is :

$$\mathbb{V}(p) = \frac{p(1-p)}{n} \quad (7)$$

And thus, the standard error is:

$$SE_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} \quad (8)$$

Thus, to compute the probability (area under curve) of the test statistic, we use the z-score:

$$z^* = \frac{p - p_0}{SE_p} \quad (9)$$

which is normally distributed.

What is a *p*-value?

Definition

The *p*-value is the smallest level of significance at which H_0 would be rejected when a specified test procedure is used on a given dataset.
Equivalently,this is the minimum probability of a Type I error.

Motivating the usage of *p*-values

Example 3: Nicotine content

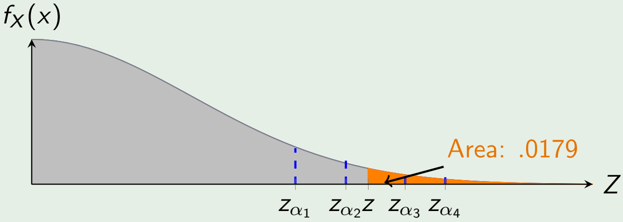
Based on data from a sample of cigarettes, the Z statistic is $z = 2.10$. You want to verify if the true nicotine content (measured in proportion of tobacco weight) is $p = .015$ (H_0) versus the alternative hypothesis that is greater: $H_1 : p > .015$).
This is an **upper-tailed** hypothesis test.
What are your conclusions from testing at the following significance levels:

- $\alpha_1 = 0.05$
- $\alpha_1 = 0.025$
- $\alpha_1 = 0.01$
- $\alpha_1 = 0.005$

Motivating the usage of p-values

Example 3: Nicotine content (cont.)

The p-value $1 - \Phi(2.10)$ (area to the right of z) $\therefore p = 1 - 0.9821 = 0.0179$.



Your conclusions are as follows:

Level of significance α	Rejection Region	Conclusion
$\alpha_1 = 0.05$	$z \geq 1.645$	Reject H_0
$\alpha_2 = 0.025$	$z \geq 1.96$	Reject H_0
$\alpha_3 = 0.01$	$z \geq 2.33$	Fail to reject H_0
$\alpha_4 = 0.005$	$z \geq 2.58$	Fail to reject H_0

Usefulness of *p*-value

- Provides more information about the strength of a test
- Indicates the smallest level at which the data is significant
- Can be compared with α irrespective of which type of test was used

Alternative definition

The *p*-value is the probability of obtaining a test statistic value at least as contradictory to H_0 as the value that actually resulted. The smaller the *p*-value, the more contradictory are the data to H_0.

Hypothesis testing with the *p*-value

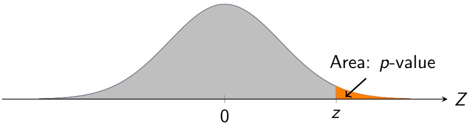
Step 1. Formulate your hypotheses

Step 2. Determine the *p*-value from the test statistic

Step 3. Conclude the test based on a chosen level of significance:

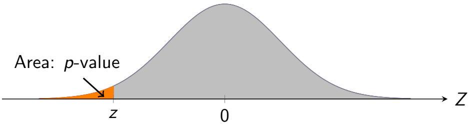
1. *p*-value $\leq \alpha \implies$ reject H_0 at level α.
2. *p*-value $> \alpha \implies$ do not reject H_0 at level α.

p-value for *z* tests



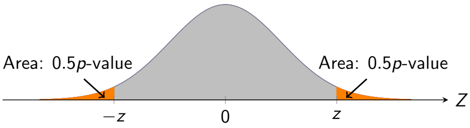
p-value: area in upper tail

$$p = 1 - \Phi(z) \qquad (10)$$



p-value: area in lower tail

$$p = \Phi(z) \qquad (11)$$



p-value: sum of area in both tails

$$p = 2(1 - \Phi(|z|)) \qquad (12)$$

Hypothesis testing using p-value approach

Example 4: Getting enough sleep (OS 5.21)

400 students were randomly sampled from a large university, and 289 said they did not get enough sleep. Conduct a hypothesis test to check whether this represents a statistically significant difference from 50%, and use a significance level of 0.01

Step 1. Parameter of interest: p (proportion of students not getting enough sleep)

Step 2. Null hypothesis: $H_0 : p = 289/400 = .723$.

Step 3. Alternative hypothesis: $H_1 : p \neq .723$.

Step 4. Formula for test statistic value: $z = \frac{p - p_0}{SE_p}$

Hypothesis testing using p-value approach

Example 4: Getting enough sleep (cont.)

Step 5. Calculate test statistic value:

$$z = \frac{.723 - .5}{\sqrt{.5(.5)/400}} = 8.92$$

Step 6. Determine p-value (two-tailed test):

$$p\text{-value} = 2(1 - \Phi(8.92)) = 0.0$$

Step 7. Conclude:
Using a significance level of 0.01, we reject H_0 since $0.0204 > 0.01$. Thus, at the 1% significance level, there is sufficient evidence to conclude that true proportion differs from the target value of 0.5.

Recap of this lecture

- Definition of hypothesis testing
  - Null hypothesis (default/expected outcome) H_0
  - Alternate hypothesis (what we want to test/support; research hypothesis) H_1 or H_A
  - One-tailed or two-tailed
- Types of errors:
  - Type I: false positive
  - Type II: false negative
- Test statistic:
  - Sample proportion with independent observations and large enough sample size (normal distribution); Z-statistic:

$$z^* = \frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \tag{13}$$

- The p-value is the minimum probability of a Type I error.
  - Upper-tailed test: $p - \text{value} = 1 - \Phi(z)$; MATLAB: `normcdf(z, 'upper')`
  - Lower-tailed test: $p - \text{value} = \Phi(z)$; MATLAB: `normcdf(z)`
  - Two-tailed test: $p - \text{value} = 2(1 - \Phi(|z|))$; MATLAB: `2 * normcdf(abs(z))`