CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 3E: The Poisson Distribution

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Outline

1. Introduction
2. The Poisson distribution
3. Examples
4. Outlook

Recap of Lecture 3b: Binomial distribution

The binomial distribution governs Bernoulli sequences.

- **Bernoulli sequence:**
  - Fixed number of statistically independent trials n
  - Each trial has only two possible outcomes: occurrence (probability p) or nonoccurrence (probability $q = 1 - p$)
  - Probability of occurrence p is constant in each trial
- Binomial distribution $X \sim \text{Bin}(n, p)$:
  - **Mean**: $\mathbb{E}(X) = np$
  - **Variance**: $\mathbb{V}(X) = npq = np(1 - p)$
  - **PMF**:

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \tag{1}$$

  - **CDF**:

$$P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1 - p)^{n-k} \tag{2}$$

Motivation

Consider the following situations:

- Text messages arrive randomly on your cellphone at various times during the day. How might you compute the probability that a given number of texts will arrive in an hour?
- Accidents happen to occur randomly in time and space along Rte 116. What is the probability that 3 accidents will occur tomorrow?
- Students seem to join the Zoom meeting for our class between 12:30 and 1:15 PM in no particular order. How would you determine the probability that between 5 and 10 students will join the Zoom meeting between 1:00 and 1:01 PM?

The above are all **Poisson processes** and the respective probabilities can be modeled by the **Poisson distribution**

Poisson process

Key assumptions:

- **Randomness:** An event can occur at random and at any instant of time or any point in space
- **Independence:** The occurrence of an event in a given time (or space) interval is statistically independent of that in any other non-overlapping interval
- **Constant rate:** The number of occurrences of an event in time interval t can be given by λt, where λ is the mean rate of occurrence of the event
- **Non-overlapping events:** The probability of two or more occurrences in t is negligible (i.e. only one event occurs each time).

Further examples of Poisson processes

- Breakdowns of a computer system over time
- Number of accidents in an industrial facility over time
- Customer arrivals at a bike store in a given morning

Poisson distribution: PMF and CDF

If X has a Poisson distribution with rate parameter λ, i.e. $X \sim \text{Poisson}(\lambda)$, its PMF is:

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x \geq 0 \tag{3}$$

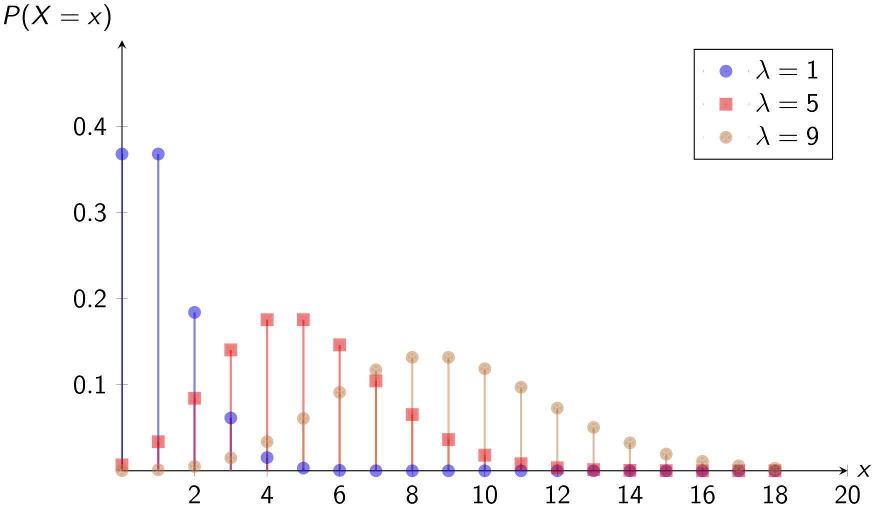
where: $\lambda \equiv$ expected/mean number of occurrences in unit time interval.

The CDF of X is given by:

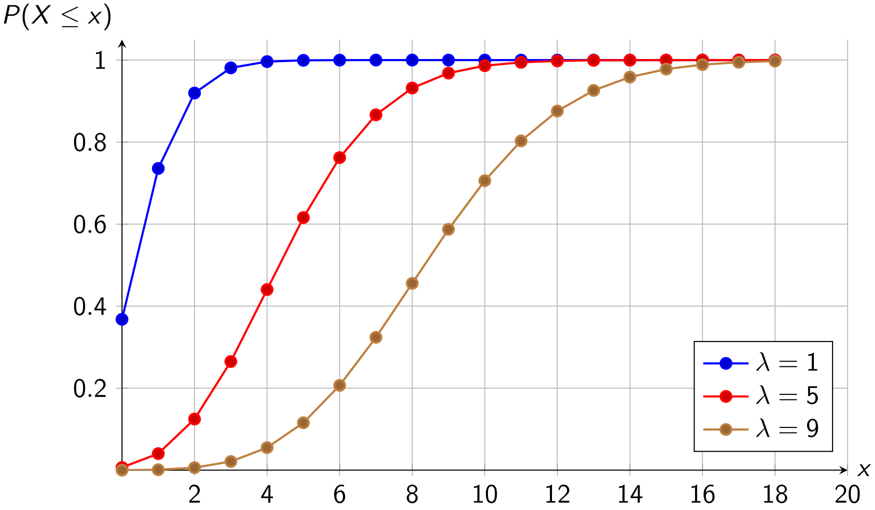
$$P(X \leq x) = \sum_{k=0}^{x} \frac{\lambda^k}{k!} e^{-\lambda} \tag{4}$$

Poisson distribution: PMF

PMF of $X \sim \text{Poisson}(\lambda)$



Poisson distribution: CDF



Mean and variance

The mean $\mathbb{E}(X)$ and variance $\mathbb{V}(X)$ of a Poisson distributed random variable are equivalent

$$\mathbb{E}(X) = \mathbb{V}(X) = \sigma_X^2 = \lambda \tag{5}$$

The standard deviation is $\sigma(X) = \sqrt{\mathbb{V}(X)} = \sqrt{\lambda}$

Computing Poisson probabilities in Python

- Load: `from scipy.stats import poisson`
- Compute the PMF $P(X = x)$ using `poisson.pmf(x, lambda)`
- Similarly, you can find the CDF $P(X \leq x)$ as `poisson.cdf(x, lambda)`
- Note that $P(X > x)$ (survival function) is obtained by
  `1 − poisson.cdf(x, lambda)`
  or
  `poisson.sf(x, lambda) #sf = survival function = 1 − cdf`

Example 1: Amherst Coffee customer arrivals

Starting at 7 AM, customers arrive Amherst Coffee according to a Poisson process at the rate of 30 customers per hour. Find the (a) probability that **exactly 40** customers arrive between 10 and 11 AM; (b) probability that **no more than 40** customers arrive between 11 AM and 12 PM; (c) **uncertainty** in the rate parameter.

Solution

The number of customers arriving within an hour $X \sim \text{Poisson}(\lambda = 30)$.

(a) The desired probability is given by:

$$P(X = 40) = \frac{30^{40}}{40!} e^{-30} = \texttt{poisson.pmf(40,30)} = \boxed{0.0139}$$

(b) The desired probability is given by:

$$P(X \leq 40) = \sum_{k=0}^{40} \frac{30^k}{k!} e^{-30} = \texttt{poisson.cdf(40,30)} = \boxed{0.9677}$$

Example 1: Amherst Coffee customer arrivals (cont.)

Solution (cont.)

(c) The uncertainty is given by the standard deviation:

$$\sigma_X = \sqrt{\lambda} = \sqrt{30} = \boxed{5.477}$$

Example 2: Aircraft arrivals at airport

Suppose small aircraft arrive at a certain airport according to a Poisson process with a rate $\lambda = 8/\text{hr}$.

(a) What is the probability that exactly 6 small aircraft arrive during a 1-hr period?

(b) What is the probability that at least 10 small aircraft arrive during a 1-hr period?

(c) How many aircraft do you expect to arrive during a 90-minute period?

Example 2: Aircraft arrivals at airport (cont.)

(a) What is the probability that exactly 6 small aircraft arrive during a 1-hr period?

Solution

$$P(X_{1hr} = 6) \quad = \quad \frac{\lambda^x e^{-\lambda}}{x!} = \frac{8^6 e^{-8}}{6!} = \texttt{poisson.pmf(6,8)} = \boxed{0.1221}$$

(b) What is the probability that at least 10 small aircraft arrive during a 1-hr period?

Solution

The desired probability is given by:

$$\begin{aligned} P(X_{1hr} > 9) &= 1 - P(X_{1hr} \leq 9) \\ &= 1 - \sum_{x=0}^{9} \frac{\lambda^x e^{-\lambda}}{x!} = \texttt{1 - poisson.cdf(9,8)} \\ &= 1 - 0.7166 = \boxed{0.2834} \end{aligned}$$

Manipulating the rate parameter

- The rate parameter λ is given by the mean number of occurrences per unit time.

- If we are to find the probability of a certain number of events within a different time interval T', where

$$T' = T \times t \tag{6}$$

- Then, we can find the new rate parameter by multiplying λ by the appropriate number of time units t, such that

$$\lambda' = \lambda t \tag{7}$$

- And thus:

$$P(X_t = x) = \frac{(\lambda t)^x}{x!} e^{-(\lambda t)} = \frac{(\lambda')^x}{x!} e^{-(\lambda')} \tag{8}$$

Example 2: Aircraft arrivals at airport (cont.)

(c) How many aircraft do you expect to arrive during a 90-minute period?

Solution

Expected arrivals within 90 minutes $= 8/\text{hr} \times 1.5 \text{ hrs} = 12$.

Example 3: Amherst Coffee customer arrivals (revisited)

Starting at 7 AM, customers arrive Amherst Coffee according to a Poisson process at the rate of 30 customers per hour. Now find the probability more than 65 customers arrive between 10 and 12 PM.

Solution

The probability is now desired for an interval twice as long. So we compute a new rate parameter for 2-hr interval:

$$\lambda^* = 30 \times 2 = 60 \text{ (per 2 hrs)}$$

The desired probability is thus given by:

$$\begin{aligned} P(X_{2hr} > 65) &= 1 - P(X_{2hr} \leq 65) = \texttt{1 - poisson.cdf(65,60)} \\ &\equiv \texttt{poisson.sf(65,60)} = \boxed{0.2355} \end{aligned}$$

Example 4: Text messages

A student receives text messages starting at 10 AM at the rate of 10 texts per hour according to a Poisson process. Find the probability that they will recieve exactly 18 texts by noon and 70 texts by 5 PM.

Solution

- The student receives 18 texts in the first 2 hrs and then $70 - 18 = 52$ texts in the next 5 hrs (5:00 PM).
- The events $X_{12} = 18$ and $X_5 = 52$ are independent within the specified time intervals.

Thus:

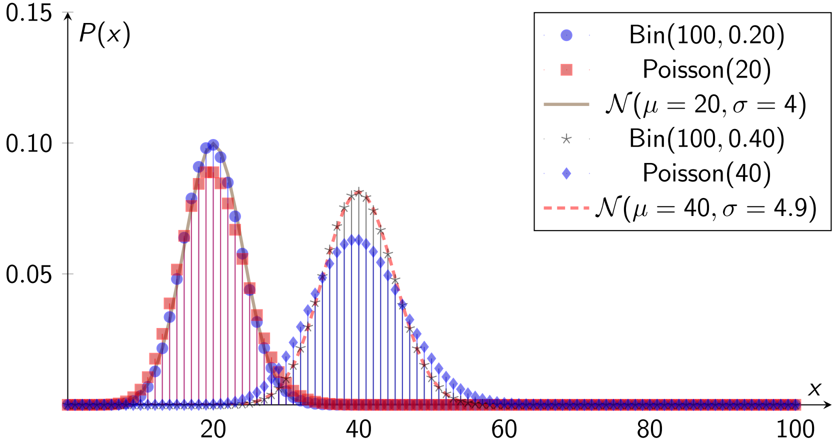
$$\begin{aligned}
P(X_{12} = 18 \cap X_5 = 52) &= P(X_{12} = 18) \times P(X_5 = 52) \\
&= \frac{(10(2))^{18}}{18!} e^{-10(2)} \times \frac{(10(5))^{52}}{52!} e^{-10(5)} \\
&= \texttt{poisson.pmf(18,20) * poisson.pmf(52,50)} \\
&= \boxed{0.0045}
\end{aligned}$$

Poisson distribution as limit of binomial distribution

For a binomially distributed r.v. X with n trials, as $n \to \infty$ and $p \to 0$, then

$$X \sim \text{Bin}(n, p) \to X \sim \text{Poisson}(\lambda) \tag{9}$$

where $\lambda = np$.



Recap

- The Poisson distribution is used to model the probability that a number of independent events occur within a fixed time interval (or within a finite space)
- Such events are described as **Poisson processes**
- The **PMF** of a Poisson random variable with rate parameter λ is given by:

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \tag{10}$$

  where $x \geq 0$
- The **mean** and **variance** of a Poisson random variable are equal:

$$\mathbb{E}(X) = \mathbb{V}(X) = \lambda \tag{11}$$

- Recall that the **standard deviation** is simply the square root of the variance:

$$\sigma(X) = \sqrt{\lambda} \tag{12}$$

Reading

- Open Intro Statistics Section 4.5