CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture M2c: Conditional Probability and Bayes' Theorem

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Outline

Recap from Lecture 2b: Theory of Probability

- Three axioms of probability:

$$P(E) \geq 0 \quad \text{and} \quad P(E) \leq 1$$
$$P(S) = 1$$
$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n) \ \text{(Mutually exclusive)}$$

- Addition rule: $P(A \cup B) = P(A) + P(B) - P(AB)$
- Addition rule for mutually exclusive events: $P(A \cup B) = P(A) + P(B)$ (Axiom 3)
- Counting
  - Fundamental principle of counting: number of outcomes for $1, \ldots, k$ events, each with $n_1, \ldots, n_k$ possibilities is $n_1 \times \cdots \times n_k$
  - Permutations of n objects: $n! = n(n-1)(n-2)\cdots(2)(1)$
  - Permutations (arrangements) of a subset of k items chosen from set of n items: $n!/(n-k)!$
  - Combinations (distinct; order not important) of group of k items chosen from set of n items: $n!/(k!(n-k)!)$

Objectives of today's lecture

- Understand conditional probability
- Grasp the concept of statistical independence
- Apply the multiplication rule and understand its relation to conditional probability
- Understand total probability
- Understand Bayes' Theorem and learn how to apply it to solving inverse probability problems

Conditional probability

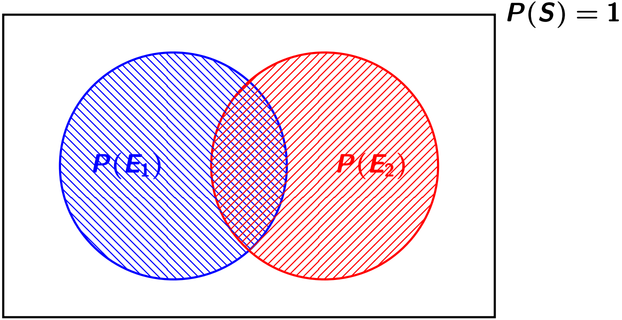
Definition

The probability of an event E_1 is conditional upon another event E_2 if the occurrence of E_1 is dependent on the occurrence of E_2:

$$
\begin{aligned}
P(E_1|E_2) &= \text{the conditional probability of } E_1 \text{ given } E_2 \\
&\qquad \text{(OR the probability of } E_1 \text{ conditioned on } E_2\text{)} \\
P(E_2|E_1) &= \text{the conditional probability of } E_2 \text{ given } E_1
\end{aligned}
$$

Conditional probability (cont.)

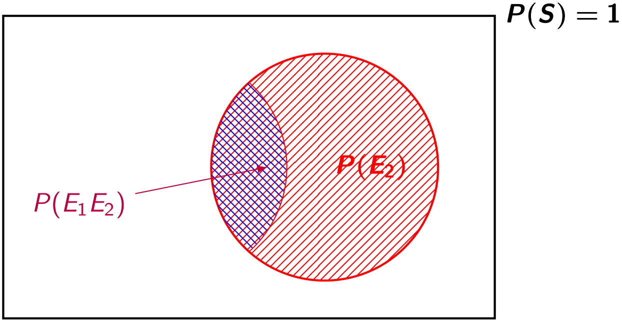
We can think of $P(E_1|E_2)$ as the probability of realizing sample points of E_1 within the subsample space of E_2.



Conditional probability (cont.)

Thus,

$$P(E_1|E_2) = \frac{P(E_1E_2)}{P(E_2)} \tag{1}$$



Useful conditional probability relations

$$P(A|S) = P(A) \tag{2}$$

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{3}$$

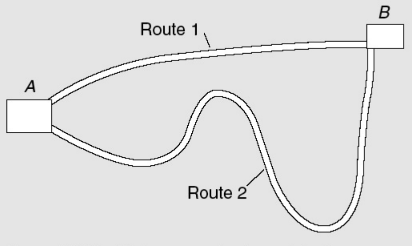
$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(AB)}{P(A)} \tag{4}$$

$$P(\overline{A}|B) = 1 - P(A|B) \tag{5}$$

- A useful mnemonic: think of "|" as the division sign.
  - This means that the event to the right is what goes in the denominator.
  - The numerator is simply the intersection of both events

Example 1: Highway routes

There are two routes from City A to City B. One or both routes may be closed due to severe weather.
We denote E_1 as the event Route 1 is open, and E_2 as the event Route 2 is open.



During severe weather, the probabilities the routes will be open are:

$$P(E_1) = 0.75 \quad P(E_2) = 0.50 \quad P(E_1 E_2) = 0.40$$

(a) What is probability that Route 1 is open during a storm given that Route 2 is also open?

(b) What is probability that Route 2 is open during a storm conditioned on the event that Route 1 is also open?

(c) What is probability that Route 1 is closed during a storm given that Route 2 is also closed?

Example 1: Highway routes (cont.)

$$P(E_1) = 0.75 \quad P(E_2) = 0.50 \quad P(E_1 E_2) = 0.40$$

(a) The probability that Route 1 is open (E_1) during a storm given that Route 2 is also open (E_2) is denoted as:

$$\begin{aligned} P(E_1|E_2) &= \frac{P(E_1 E_2)}{P(E_2)} \\ &= \frac{0.40}{0.50} = \boxed{0.80} \end{aligned}$$

Example 1: Highway routes (cont.)

$$P(E_1) = 0.75 \quad P(E_2) = 0.50 \quad P(E_1 E_2) = 0.40$$

(b) The probability that Route 2 is open (E_2) during a storm conditioned on the event that Route 1 is also open (E_1) is denoted as:

$$\begin{aligned} P(E_2|E_1) &= \frac{P(E_2 E_1)}{P(E_1)} \\ &= \frac{0.40}{0.75} = \boxed{0.53} \end{aligned}$$

Example 1: Highway routes (cont.)

$$P(E_1) = 0.75 \quad P(E_2) = 0.50 \quad P(E_1 E_2) = 0.40$$

(c) The probability that Route 1 is closed ($\overline{E}_1$) during a storm given that Route 2 is also closed ($\overline{E}_2$) is denoted as:

$$P(\overline{E_1}|\overline{E_2}) = \frac{P(\overline{E_1}\,\overline{E_2})}{P(\overline{E_2})}$$

$$\begin{aligned} P(\overline{E_1}\,\overline{E_2}) &= 1 - P(\overline{\overline{E_1}\,\overline{E_2}}) \quad \text{complementary events} \\ &= 1 - P(E_1 \cup E_2) \quad \text{De Morgan's Rule} \\ &= 1 - [P(E_1) + P(E_2) - P(E_1 E_2)] \quad \text{addition rule} \\ &= 1 - (0.75 + 0.50 - 0.40) \\ &= 1 - 0.85 = 0.15 \end{aligned}$$

Thus, we obtain

$$P(\overline{E_1}|\overline{E_2}) = \frac{P(\overline{E_1}\,\overline{E_2})}{P(\overline{E_2})} = \frac{0.15}{0.50} = \boxed{0.30}$$

The addition rule

Conditioning equally applies to probabilities of events on the same subsample space.

$$P(A \cup B) = P(A) + P(B) - P(AB) \tag{6}$$

$$P(A \cup B|E) = P(A|E) + P(B|E) - P(AB|E) \tag{7}$$

Statistically independent events

Statistical independence

Two events are statistically independent if the occurence of one does not depend on the occurence or non-occurence of the other:

$$P(E_1|E_2) = P(E_1) \tag{8}$$

$$P(E_2|E_1) = P(E_2) \tag{9}$$

Examples of independent events

- The outcomes of a die rolled twice in succession. The result of the first roll does not affect the result of the other.
- The event that it will be cloudy in Amherst tomorrow and the event that the number of births in California tomorrow will increase compared to that of today. (Knowledge of one event cannot improve prediction of the other.)
- The ages of a random sample of Pioneer Valley residents
- Generally, all elements of a random sample are assumed independent

Example 2: Rolling two dice

Let X and Y represent the outcomes of rolling two dice.

(a) What is the probability that the first die X is 1?

$$P(X = 1) = \frac{1}{6}$$

(b) What is the probability that both X and Y are 1?

The outcome of X has no bearing on the outcome of Y. Thus:

$$\begin{aligned} P(X = 1 \cap Y = 1) &= P(X = 1) \times P(Y = 1) \\ &= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \end{aligned}$$

Example 2: Rolling two dice (cont.)

(c) Use the conditional probability formula to find $P(Y = 1|X = 1)$

$$
\begin{aligned}
P(Y = 1|X = 1) &= \frac{P(Y = 1 \cap Y = 1)}{P(X = 1)} \\
&= \frac{P(Y = 1) \times \cancelto{1}{P(X = 1)}}{\cancelto{1}{P(X = 1)}} \\
&= P(Y = 1)
\end{aligned}
$$

(d) Why is $P(Y = 1|X = 1) = P(Y = 1)$?

X and Y are independent events.

The multiplication rule

The probability of the intersection of two events is

$$P(E_1 E_2) = P(E_1|E_2)P(E_2) \tag{10}$$

Equivalently:

$$P(AB) = P(B|A)P(A) \tag{11}$$

Statistically independent events

IF A and B are statistically independent, the multiplication rule becomes:

$$P(AB) = P(A)P(B) \tag{12}$$

That is, the joint probability of two statistically independent events is the product of their individual probabilities.

The multiplication rule: three events

The probability of the joint occurrence of three events is:

$$\begin{aligned} P(ABC) &= P(A|BC)P(BC) \\ &= P(A|BC)P(B|C)P(C) \end{aligned} \tag{13}$$

Statistically independent events

If the three events are statistically independent, then:

$$P(ABC) = P(A)P(B)P(C) \tag{14}$$

The multiplication rule: conditional probabilities

Equally applies to probabilities of events conditioned on the same subsample space.

$$P(AB|E) = P(A|B|E)P(B|E) \tag{15}$$

Statistically independent events

If A and B are statistically independent, the multiplication rule for two events conditioned on the same space becomes:

$$P(AB|E) = P(A|E)P(B|E) \tag{16}$$

Example 3: Airline industry strikes

The airline industry in a certain country is subject to labor strikes by the pilots (event A), mechanics (event B) or flight attendants (event C).

The probability of strikes by each of the individual groups in the next 3 years is given by:

$$P(A) = 0.03 \quad P(B) = 0.05 \quad P(C) = 0.06$$

Assuming that events A, B and C are statistically independent, find

(a) the probability that all groups will strike in the next 3 years

(b) the probability of a labor strike in the airline industry in the next 3 years.

Example 3: Airline industry strikes (cont.)

$$P(A) = 0.03 \quad P(B) = 0.05 \quad P(C) = 0.06$$

(a) The probability that all 3 groups will strike in the next 3 years is given by:

$$\begin{aligned} P(ABC) &= P(A)P(B)P(C) \quad \text{statistical independence} \\ &= 0.03(0.05)(0.06) \\ &= 0.00009 = \boxed{9.0 \times 10^{-5}} \end{aligned}$$

Example 3: Airline industry strikes (cont.)

$$P(A) = 0.03 \quad P(B) = 0.05 \quad P(C) = 0.06$$

(b) The probability of a labor strike is the probability that any combination of the groups will strike.

(Step 1) Formulate the desired probability:

$$P(A \cup B \cup C)$$

(Step 2) Use the complement rule, de Morgan's rule and multiplication rule for statistically independent events:

$$\begin{aligned} P(A \cup B \cup C) &= 1 - P(\overline{A \cup B \cup C}) \quad \text{complement rule} \\ &= 1 - P(\overline{A}\,\overline{B}\,\overline{C}) \quad \text{de Morgan's rule} \\ &= 1 - P(\overline{A})P(\overline{B})P(\overline{C}) \quad \text{statistical independence} \\ &= 1 - (1 - 0.03)(1 - 0.05)(1 - 0.06) \\ &= 1 - (0.97)(0.95)(0.94) \\ &= \boxed{0.134} \end{aligned}$$

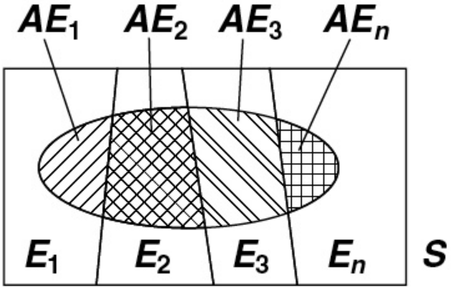
Total probability

Useful in situations where the probability of an event cannot be directly determined but its conditional probabilities are known.

Theorem of total probability

The probability of an event A conditioned on the mutually exclusive and collectively exhaustive events $E_1, E_2, \ldots, E_n$ is given by

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \cdots + P(A|E_n)P(E_n) \tag{17}$$



$$P(A) = P(AE_1) + P(AE_2) + P(AE_3) + \cdots + P(AE_n)$$

Note that:

$P(AE_1) = P(A|E_1)P(E_1)$,

etc.

Example 4: Flooding and snow accumulation

The flooding of a river in the spring season (event F) will depend on the accumulation of snow in the mountain during the past winter.The accumulation of snow may be heavy (H), normal (N) or light (L).

Given:

$$P(F|H) = 0.90 \quad P(F|N) = 0.40 \quad P(F|L) = 0.10$$

and in the winter

$$P(H) = 0.20 \quad P(N) = 0.50 \quad P(L) = 0.30$$

Find the probability of flooding in the river during the spring season $P(F)$.

Example 4: Flooding and snow accumulation (cont.)

We use the theorem of total probability:

$$\begin{aligned} P(F) &= P(F|H)P(H) + P(F|N)P(N) + P(F|L)P(L) \\ &= 0.90(0.20) + 0.40(0.50) + 0.10(0.30) \\ &= 0.41 \end{aligned}$$

Derivation of Bayes' theorem

Recall from the multiplication rule that:

$$P(AB) = P(A|B)P(B) \tag{18}$$

Equivalently:

$$P(AB) = P(B|A)P(A) \tag{19}$$

We combine both equations to obtain:

$$P(A|B)P(B) = P(B|A)P(A) \tag{20}$$

Then, we obtain the **inverse probability** of the conditioning event:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \tag{21}$$

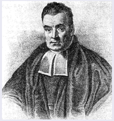
This is in essence Bayes' Theorem

Bayes' theorem

Bayes' Theorem allows for the computation of an inverse probability, e.g. given $P(E|A)$, can we find $P(A|E)$?

$$P(A_i|E) = \frac{P(E|A_i)P(A_i)}{\sum_{j=1}^{n} P(E|A_j)P(A_j)} = \frac{P(E|A_i)P(A_i)}{P(E)} \tag{22}$$

- **posterior probability:** $P(A_i|E)$
- **likelihood:** $P(E|A_i)$
- **prior:** $P(A_i)$
- **evidence (total probability):** $P(E)$



Rev. Thomas Bayes (1701-61)

If the event E can be conditioned on only two events A and B, then:

$$P(A|E) = \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B)} \tag{23}$$

$$P(B|E) = \frac{P(E|B)P(B)}{P(E|A)P(A) + P(E|B)P(B)} \tag{24}$$

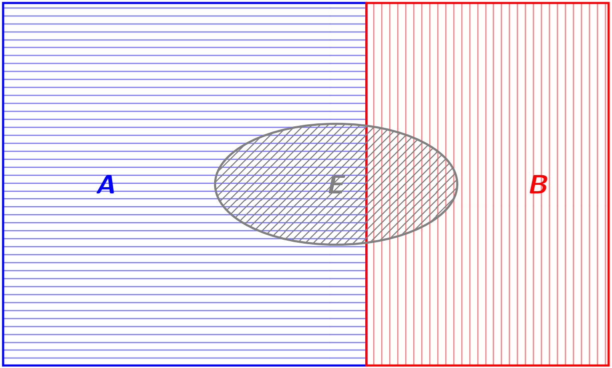
Example 5: Construction supplies

Aggregates for the construction of a reinforced concrete building are supplied by two companies. Company a delivers 600 truckloads a day while Company b delivers 400 truckloads a day. From prior experience, 3% of Company a's material is expected to be substandard while 1% of Company b's material is expected to be susbstandard.
We define:

$$
\begin{aligned}
A &= \text{aggregates supplied by Company } a \\
B &= \text{aggregates supplied by Company } b \\
E &= \text{aggregates are substandard}
\end{aligned}
$$

Example 5: Construction supplies (cont.)

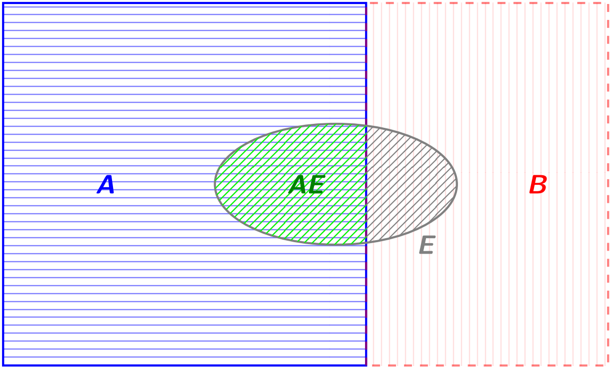
(a) Draw a Venn diagram and convince yourself that
$P(A) = 0.60, P(B) = 0.40, P(E|A) = 0.03, P(E|B) = 0.01$



Example 5: Construction supplies (cont.)

$$P(A) = 0.60, P(B) = 0.40, P(E|A) = 0.03, P(E|B) = 0.01$$

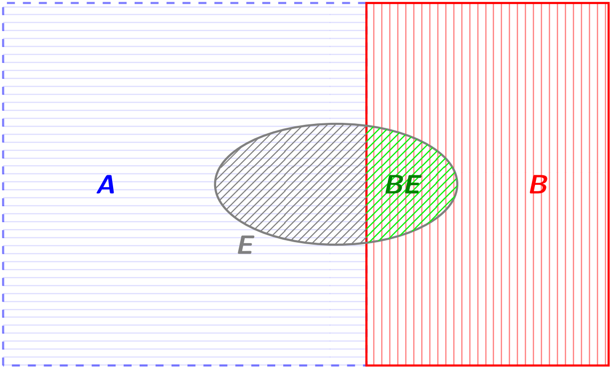
(a) $P(E|A) = \frac{P(EA)}{P(A)}$



Example 5: Construction supplies (cont.)

$$P(A) = 0.60, P(B) = 0.40, P(E|A) = 0.03, P(E|B) = 0.01$$

(a) $P(E|B) = \frac{P(EB)}{P(B)}$



Example 5: Construction supplies (cont.)

$$P(A) = 0.60, P(B) = 0.40, P(E|A) = 0.03, P(E|B) = 0.01$$

(b) Show that $P(E) = 0.022$

We use the Theorem of Total Probability:

$$\begin{aligned} P(E) &= P(E|A)P(A) + P(E|B)P(B) \\ &= (0.03)(0.6) + (0.01)(0.4) \\ &= 0.018 + 0.004 \\ &= \boxed{0.022} \end{aligned}$$

Example 5: Construction supplies (cont.)

(c) Show that the probability $P(A|E) = 0.82$ and discuss its significance

Here, we use Bayes' Theorem:

$$\begin{aligned} P(A|E) &= \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B)} \\ &= \frac{P(E|A)P(A)}{P(E)} \quad \text{Denominator: total probability} \\ &= \frac{0.03 \times 0.60}{0.022} \equiv \frac{\text{likelihood} \times \text{prior}}{\text{evidence}} \\ &= 0.818 \approx \boxed{0.82} \end{aligned}$$

$P(A|E)$ is the posterior probability of A having observed E. In other words, having prior knowledge of A (i.e. $P(A)$ and the likelihood of of E given A, we can update our knowledge of A (posterior probability) based on these observations.

Recap

- **Conditional probability**:

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{25}$$

- **Independent events**:

$$P(AB) = P(A)P(B) \tag{26}$$

Generally, the joint probability (intersection) of any number of independent events is the product of their individual probabilities:

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2)\cdots P(E_n) \tag{27}$$

- **Multiplication rule**:

$$P(AB) = P(A|B)P(B) = P(B|A)P(A) \tag{28}$$

Recap (cont.)

- Total probability:

$$P(E) = P(E|A_1)P(A_1) + P(E|A_2)P(A_2) + \cdots + P(E|A_n)P(A_n) \tag{29}$$

- Bayes' Theorem:

$$P(A_1|E) = \frac{P(E|A_1)P(A_1)}{P(E|A_1)P(A_1) + P(E|A_2)P(A_2) + \cdots + P(E|A_n)P(A_n)} \tag{30}$$