CEE 260/MIE 273: Probability and Statistics in Civil Engineering
Lecture 2b: Theory of Probability

Prof. Oke

UMass Amherst

College of Engineering

September 16, 2025

Outline

Recap from Lecture 2a

- Set theory and operations: union, intersection, complement
- Events occur within a sample space of possibilities
- The sample space can be discrete (infinite/finite) or continuous (infinite)
- Mutually exclusive events cannot jointly occur:

$$A \cap B = \varnothing, \quad \text{(if } A \text{ and } B \text{ are mutually exclusive)}$$

- The union of collectively exhaustive events yields the sample space:

$$A \cup B = S \quad \text{(if } A \text{ and } B \text{ are collectively exhaustive)}$$

- De Morgan's Rules are useful for expressing complements of unions or of intersections:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Objectives of today's lecture

- Understand the axioms of probability
- Learn to use the addition rule in computing probabilities
- Understand counting principles:
  - Fundamental principle of counting
  - Permutations
  - Combinations
- Perform data input, set operations, permutations and combinations in MATLAB

Probability

- Probability refers to the *likelihood* that an event within a sample space occurs
- This can be expressed as the relative frequency with which such an event occurs over (infinitely) many trials
- A probability can thus be given by a proportion or fraction

Notation

Given an event E, the probability that E occurs is denoted as $P(E)$.

Examples

- Assuming all outcomes are equally likely, what is the probability of rolling a "3" with a six-sided die? $\boxed{\frac{1}{6}}$
- In a certain university, students have only two degree options: BA and BSc. If the ratio of BA to BSc students on campus is 3:2, what is the probability that a student selected purely at random from a roster of all the students is signed up for a BSc? $\boxed{P(BSc) = \frac{2}{3+2} = \frac{2}{5} = 0.40}$

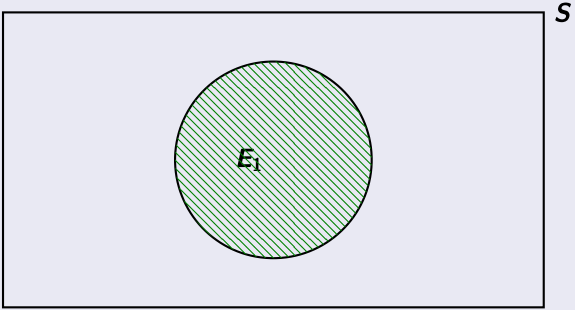
Basic axioms of probability (1)

Axiom 1

For every event E in sample space S:

$$P(E) \geq 0 \tag{1}$$

You can think of $P(E)$ as the fraction of outcomes in E relative to the sample space (S):



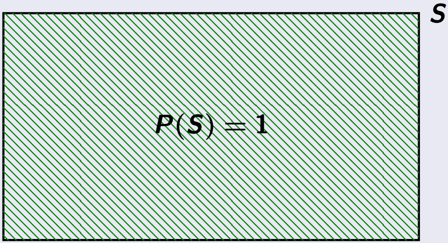
Basic axioms of probability (2)

Axiom 2

The probability of a certain event S is:

$$P(S) = 1 \tag{2}$$

A certain event occupies the entire sample space



In other words, the probability of the sample space is unity (or 1).

Basic axioms of probability (3)

Axiom 3

The probability of the union of n mutually exclusive events is given by:

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n) \tag{3}$$

Equivalently,

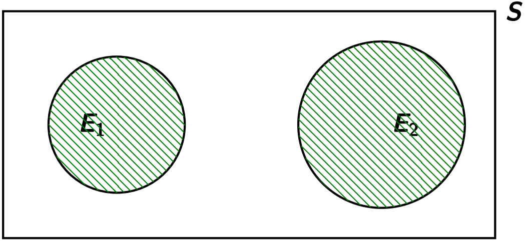
$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \tag{4}$$

where:

- $\bigcup_{i=1}^{n}$ denotes the union of sets (in this case, events E_i) indexed from 1 through n
- $\sum_{i=1}^{n}$ denotes the summation of quantities (in this case, probabilities $P(E_i)$) indexed from 1 through n

Example 1: Probability of mutually exclusive events

If E_1 and E_2 are mutually exclusive events with probabilities 0.2 and 0.3, respectively. Find $P(E_1 \cup E_2)$.



$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = 0.2 + 0.3 = \boxed{0.5}$$

Implications

- The probability of an event is always relative to that of others in the sample space
- It is therefore convenient to normalize the probability of an event to that of its sample space (i.e. 1)
- Thus

$$0 \leq P(E) \leq 1 \tag{5}$$

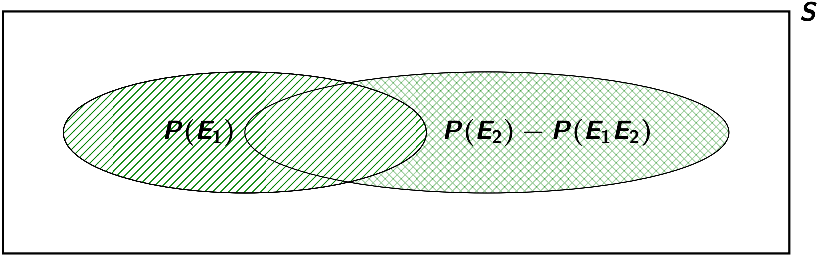
The addition rule

Generally, the probability of the union of two events E_1 and E_2 is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2) \tag{6}$$

Recall that $E_1 E_2 \equiv E_1 \cap E_2$.

The Venn diagram enables us to better visualize this.



Addition rule for mutually exclusive events

Recall: given two events E_1 and E_2:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2) \tag{7}$$

If the events E_1 and E_2, are mutually exclusive, then

$$P(E_1 E_2) = 0$$

Thus,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - 0 \tag{8}$$

which yields Axiom 3.

Complementary events

Recall that the union of an event and its complement yields the entire sample space:

$$E \cup \overline{E} = S \tag{9}$$

Thus, we have

$$P(E \cup \overline{E}) = P(S) = 1 \tag{10}$$

Given that E and $\overline{E}$ are also mutually exclusive, from Eq. 8 we obtain:

$$P(E \cup \overline{E}) = P(E) + P(\overline{E}) = 1 \tag{11}$$

$$\therefore P(\overline{E}) = 1 - P(E) \tag{12}$$

Equivalently, $P(E) = 1 - P(\overline{E})$. *Complement Rule*

Example 2: Left-turn lane design

60 observations of the number of vehicles waiting for left turns at an intersection are shown in the table.

No. of Vehicles Waiting	No. of Observations	Relative Frequency
0	4	4/60
1	16	16/60
2	20	20/60
3	14	14/60
4	3	3/60
5	2	2/60
6	1	1/60
7	0	0
8	0	0

Given the following definitions:

$$E_1 = > 2 \text{ vehicles waiting for left turns}$$
$$E_2 = \leq 4 \text{ vehicles waiting for left turns}$$

Find the following probabilities:

(a) $P(E_1)$
(b) $P(E_2)$
(c) $P(E_1 E_2)$
(d) $P(E_1 \cup E_2)$

Example 2: Left-turn lane design (cont.)

No. of Vehicles Waiting	No. of Observations	Relative Frequency
0	4	4/60
1	16	16/60
2	20	20/60
3	14	14/60
4	3	3/60
5	2	2/60
6	1	1/60
7	0	0
8	0	0

Let X be the number of vehicles.

(a) $P(E_1) = P(X > 2) = 1 - P(\overline{E_1}) = 1 - \frac{4}{60} - \frac{16}{60} - \frac{20}{60} = \frac{20}{60}$

(b) $P(E_2) = P(X \leq 4) = \frac{4}{60} + \frac{16}{60} + \frac{20}{60} + \frac{14}{60} + \frac{3}{60} = \frac{57}{60}$

(c) $P(E_1 E_2) = P(X > 2 \text{ and } X \leq 4) = P(2 < X \leq 4)$

$= P(X = 3) + P(X = 4) = \frac{14}{60} + \frac{3}{60} = \frac{17}{60}$

(d) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2) = \frac{20+57-17}{60} = 1$

How would you characterize E_1 and E_2? Collectively exhaustive events

Fundamental principle of counting

Given $1, 2, \ldots, k$ operations are to be performed, and $n_1, n_2, \ldots, n_k$ ways of performing each respective operation.

The total number of possibilities is given by

$$n_1 \times n_2 \times \cdots \times n_k$$

Example 3: Ice cream store

An ice cream store has the following options for an order (only one size is available):

- Three types of cone (cups are not available)
- Ten flavors
- Four toppings (which cannot be combined)

How many distinct ice cream orders are possible?

$$3 \times 10 \times (4 + 1) = \boxed{150}$$

(Note: you can order an ice cream without any topping, hence 5 topping possibilities

Example 4: License plate numbers

A certain state uses only digits (including "0") for its license plate numbers.

If every number must have 5 digits, how many distinct license plate numbers are available?

There are 10 digits $(0, 1, \ldots, 9)$.
The number of possible license plates is thus given by

$$10 \times 10 \times 10 \times 10 \times 10 = 10^5$$

If in 2021, the state releases a new series of plate numbers that must now end with a letter in addition to 5 preceding digits, how many distinct plates can be manufactured? Answer: $26 \times 10^5 = 2,600,000$

Permutations

A permutation is an arrangement (ordering) of objects in a collection.

Definition

The number of permutations of n objects is given by $n!$ (pronounced "n factorial"), where

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \tag{13}$$

Examples of factorials

- $2! = 2 \times 1 = 2$
- $3! = 3 \times 2 \times 1 = 6$
- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $5! = 5 \times 4! = 120$
- $0! = 1$

Example 5: Ordering numbers

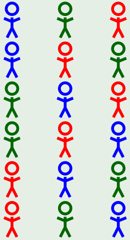
In how many ways can 3 rag dolls be arranged in a row?

The number of possibilities is given by:

$$3! = 3 \times 2 \times 1 = 6$$

Let's count them:

1
2
3
4
5
6



Permutations of a subset

The number of ways n objects can be arranged is $n!$

However, if a subset of k objects is chosen from a set of n objects, the number of permutations (ways of choosing and arranging) of this subset is given by

$$\frac{n!}{(n-k)!} \tag{14}$$

Example 6: 3-letter license plates

In a small state, plate numbers must have only 3 non-repeating characters (letters). How many permutations are there?

Method 1. There are 26 possibilities for the first character. After this, 25 options are left for the second character, and then 24 for the third. The permutations are thus: $\boxed{26 \times 25 \times 24 = 15600}$

Method 2. We can use the formula for the permutations of a subset of k items from a set of n:

$$\begin{aligned}
\frac{n!}{(n-k)!} &= \frac{26!}{(26-3)!} = \frac{26!}{(23)!} \\
&= \frac{26 \times 25 \times 24 \times \cancelto{1}{23!}}{\cancelto{1}{23!}} \\
&= 26(25)(24) = 15600
\end{aligned}$$

Permutations with repetition

Without repetition, the number of permutations of n objecst is $n!$.

However, if there are n_1 identical (or repeated) items of type 1, n_2 identical items of type 2, . . . , and n_k identical items of type k, then the number of permutations is given by

$$\frac{n!}{n_1!n_2! \times \cdots \times n_k!} \tag{15}$$

How many ways can the letters in the word "MASS" be arranged?

The letter "S" is repeated. Thus, the number of permutations are:

$$\frac{4!}{2!} = 4 \times 3 = 12$$

Combinations

A combination is distinct subset of objects selected from a collection.

Definition

The number of possible combinations of k objects chosen from a collection of n objects is given by $\binom{n}{k}$ (pronounced "n choose k"):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{16}$$

With combinations, the order is not important (in constrast to permutations).

Example 7: Choosing a basketball team

You need to form a basketball team of five. You have a group of 8 players to choose from. How many different teams can you form?

The number of combinations is given by

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!}$$
$$= \frac{8 \times 7 \times 6 \times \cancelto{1}{5!}}{\cancelto{1}{5!} \cdot 3!}$$
$$= \frac{8(7)(6)}{3(2)(1)} = \boxed{56}$$

You can form 56 different teams.

Recap

- Three **axioms** of probability:

$$\begin{aligned} \text{Ax. 1: } P(E) &\geq 0 \quad \text{and} \quad P(E) \leq 1 \\ \text{Ax. 2: } P(S) &= 1 \\ \text{Ax. 3: } P(E_1 \cup E_2 \cup \cdots \cup E_n) &= P(E_1) + P(E_2) + \cdots + P(E_n) \end{aligned}$$

- Addition rule: $P(A \cup B) = P(A) + P(B) - P(AB)$
  - For mutually exclusive events: $P(A \cup B) = P(A) + P(B) \quad \text{(Axiom 3)}$
- Counting
  - **Fundamental principle** of counting: number of outcomes for $1, \ldots, k$ events, each with $n_1, \ldots, n_k$ possibilities is $n_1 \times \cdots \times n_k$
  - **Permutations** (arrangements) of n objects: $n! = n(n-1)(n-2)\cdots(2)(1)$
  - Permutations of a **subset** of k items chosen from set of n items: $n!/(n-k)!$
  - **Combinations** (distinct; order not important) of group of k items chosen from set of n items: $n!/(k!(n-k)!)$

The addition rule: three events

The probability of the union of three events is given by

$$\begin{aligned}
P(E_1 \cup E_2 \cup E_3) &= P[(E_1 \cup E_2) \cup E_3] \\
&= P(E_1 \cup E_2) + P(E_3) - P[(E_1 \cup E_2)E_3] \quad \text{addition rule} \\
&= P(E_1) + P(E_2) - P(E_1E_2) + P(E_3) \\
&\qquad - [P(E_1E_3 \cup E_2E_3) \quad \text{addition rule; distributive} \\
&= P(E_1) + P(E_2) - P(E_1E_2) + P(E_3) \\
&\qquad - [P(E_1E_3) + P(E_2E_3) - P(E_1E_3E_2E_3) \\
&= P(E_1) + P(E_2) - P(E_1E_2) + P(E_3) \\
&\qquad - P(E_1E_3) - P(E_2E_3) + P(E_1E_2E_3) \\
\therefore \boldsymbol{P(E_1 \cup E_2 \cup E_3)} &= \boldsymbol{P(E_1) + P(E_2) + P(E_3)} \\
&\qquad \boldsymbol{- P(E_1E_2) - P(E_1E_3) - P(E_2E_3) + P(E_1E_2E_3)}
\end{aligned} \tag{17}$$

Complementary events (cont.)

What if we want to find the probability of the union of multiple events?

$$P(E_1 \cup E_2 \cup \cdots E_n) = 1 - P(\overline{E_1 \cup E_2 \cup \cdots \cup E_n}) \tag{18}$$

How would we simplify the right-hand side?
Apply de Morgan's rule:

$$\therefore P(E_1 \cup E_2 \cup \cdots E_n) = 1 - P(\overline{E_1}\,\overline{E_2} \cdots \overline{E_n}) \tag{19}$$