CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture 2a: Events and Set Operations

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Outline

Module 2: Probability

Key goals for this module:

- Understand basic set theory and operations
- Understand introductory probability theory
- Learn the fundamentals of conditional probability and Bayes' theorem

Objectives of today's lecture

- Learn the fundamentals of set theory and operations
- Understand events and sample spaces
- Use set theory to express combinations of events
- Understand the concepts of mutual exclusivity and collective exhaustivity of events

Reals, rationals, integers and natural numbers

- A **real** number is the value of a continuous quantity that can either be expressed as an infinite decimal expansion or on a number line.
  The set of real numbers is denoted by $\mathbb{R}$ and it is the superset of rational and irrational numbers.

- A **rational** number can be expressed as a fraction of two integers.
  The set of rational numbers is denoted by $\mathbb{Q}$

- The set of **integers** $\{\dots, -2, -1, 0, 1, 2, \dots\}$ is denoted by $\mathbb{Z}$.
  It is the superset of natural numbers.

- The set of **natural** numbers (used for counting) is denoted by $\mathbb{N}$.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \tag{1}$$

The symbol "$\subset$" means "is a subset of"

Key definitions

Set theory provides tools for characterizing sample spaces and thus formulating probabilistic problems.

Sample space: A collection of individual possibilities (*sample points*)

Event: A subset of the sample space

Finite set: has 1-1 correspondence with a **bounded subset** of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

Countable set: has 1-1 correspondence with a **subset** of natural numbers $\mathbb{N}$

Infinite set: has 1-1 correspondence with an **unbounded set** of real numbers $\mathbb{R}$

Uncountable set: has 1-1 correspondence with the **entire set** of real numbers $\mathbb{R}$

Sample spaces

- A sample space may be:
  1. Discrete (countable sample points—finite or infinite)
  2. Continuous (uncountable/infinite)
- All **uncountable** sets are **infinite**
  - e.g. the set of real numbers $\mathbb{R}$
- A continuous sample space is infinite
- A *countably infinite* set is both countable and infinite, e.g.
  - the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$
  - the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

	Countable	**Uncountable**
Finite	Discrete	—
Infinite	Discrete	Continuous

Example 1: Characterizing sample spaces

Characterize the following as *finite discrete*, *infinite discrete* or *continuous*:

- Number of flaws in a given length of welding. *infinite discrete*
- Total number of flights departing from Bradley International Airport in a given day. *finite discrete*
- Arrival time of passenger in metro station relative to departure of last train $[0, T]$, where T is the interval between two consecutive trains. *continuous*
- The number of days in a year with potentially measurable precipitation in Seattle. *finite discrete*

Set notation

Symbol	Meaning
$\cup$	union
$\cap$	intersection
$\supset$	proper superset
$\subset$	proper subset
$\supseteq$	superset or equal to
$\subseteq$	subset or equal to
$\overline{E}$ or E^c	complement of E
$\varnothing$	empty/null set

Set equality

Given a set A and sample space S:

$$A \cup \varnothing = A \tag{2}$$
$$A \cap \varnothing = \varnothing \tag{3}$$
$$A \cup A = A \tag{4}$$
$$A \cap A = A \tag{5}$$
$$A \cup S = S \tag{6}$$
$$A \cap S = A \tag{7}$$

- The union or intersection of a set with itself yields the same set
- The intersection of a set with the empty set yields the empty set
- ...

Set properties: union and intersection

Commutative property

$$A \cup B = B \cup A \tag{8}$$
$$A \cap B = B \cap A \tag{9}$$

Associative property

$$(A \cup B) \cup C = A \cup (B \cup C) \tag{10}$$
$$(AB)C = A(BC) \tag{11}$$

Distributive property

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \tag{12}$$
$$(AB) \cup C = (A \cup C) \cap (B \cup C) \tag{13}$$

Set properties: complements

Given an event E and a sample space S:

$$E \cup \overline{E} = S \tag{14}$$

$$E \cap \overline{E} = \varnothing \tag{15}$$

$$\overline{\overline{E}} = E \tag{16}$$

Example 2: Set operations

Given the sets:

$$\begin{aligned} A &= \{1,3,8,10\} \\ B &= \{0,2,5,7,10\} \end{aligned}$$

Find:

1. $A \cup B$ ("A union B"):

$$A \cup B = \{0,1,2,3,5,7,8,10\}$$

2. $A \cap B$ ("A intersection B"):

$$A \cap B = \{10\}$$

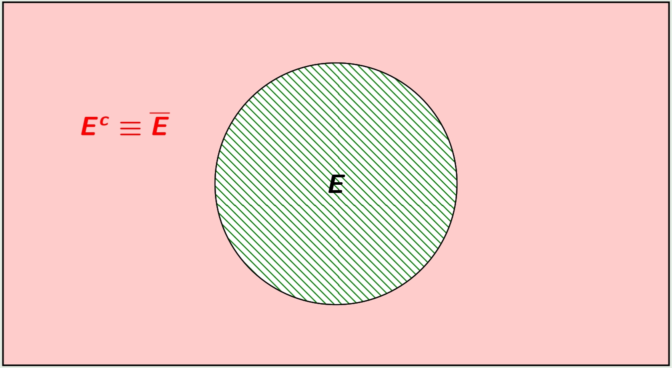
3. If the sample space is given by the integers in the interval $[0,10]$, find $(A \cup B)^c$:

$$(A \cup B)^c = \{4,6,9\}$$

Venn diagrams

An approach for visualizing sets (sample spaces) and analyzing events.

Venn diagram showing an event E and its complement $\overline{E}$



Activity: Pizza Preference Survey

Setup

- A = students who like pepperoni
- B = students who like mushrooms
- C = students who like pineapple

Activity

- Stand and sort yourselves physically into regions of the room
- Start with just sets A and B, creating a human Venn diagram
- Add set C and watch the complexity emerge
- Count each region and calculate: $|A \cup B|$, $|A \cap B|$, $|A^c|$, $|B^c|$, $|C^c|$.

Events

An event E contains one or more sample points within a sample space S

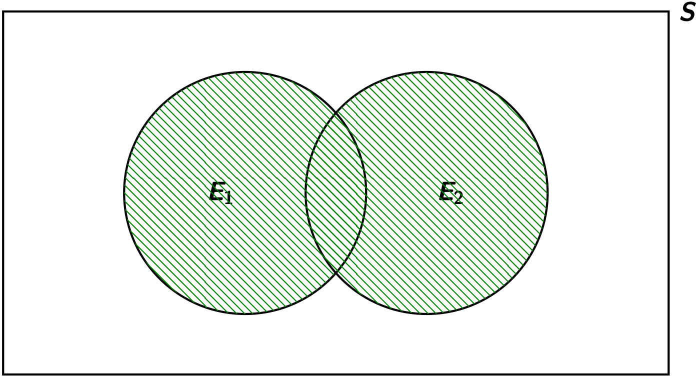
- Events can be derived from other events by union or by intersection
- An impossible event is an empty set $\varnothing$
- A certain event contains all the sample points in a sample space
  What is the probability of a certain event?
- A complementary event $\overline{E}$ of an event E contains all the sample points in S not in E

$$\overline{E} = S \setminus E \tag{17}$$

Union

Definition

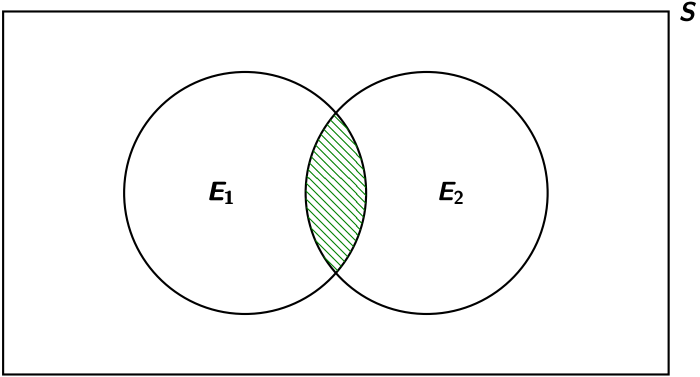
The union of two events E_1 and E_2 (denoted $E_1 \cup E_2$) is the occurrence of E_1 or E_2 or both.



Intersection

Definition

The intersection of two events E_1 and E_2 (denoted $E_1 \cap E_2$ or $E_1 E_2$) is the joint occurrence of E_1 and E_2.

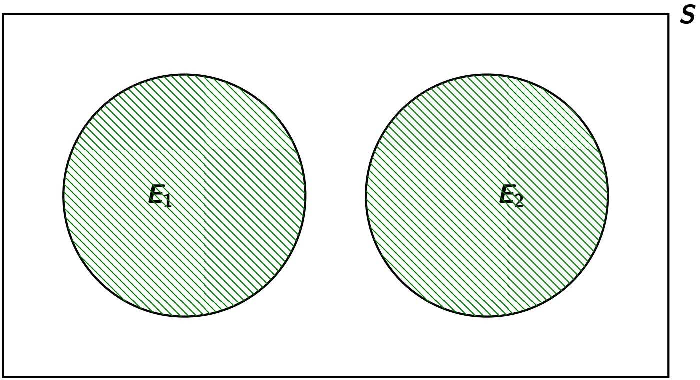


Mutually exclusive events

Definition

Two or more events are mutually exclusive if the occurrence of one event precludes the occurrence of any or all of the others:

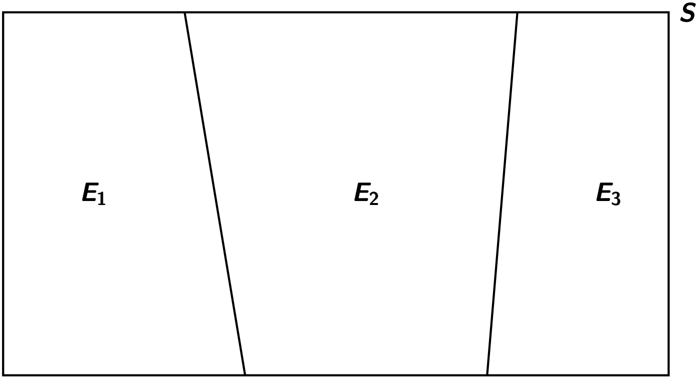
$$E_1 \cap E_2 = \varnothing \tag{18}$$



Collectively exhaustive events

Definition

A group of events are collectively exhaustive if their union is equal to the sample space containing the events. $E_1 \cup E_2 \cup E_3 = S$



Example 3: Bidding for projects

Two construction companies *a* and *b* are bidding for projects. Define A as the event that Company *a* wins a bid, and B likewise for *b*. Sketch the Venn diagrams and characterize the following events:

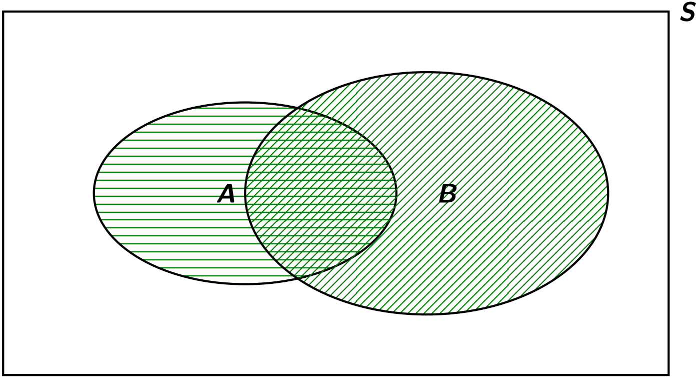
(i) Company *a* submitting a bid for one project and Company *b* submitting a bid for another project

(ii) Companies *a* and *b* submitting bids for the same project.

(iii) Company *a* and company *b* are the only bidders for the single project

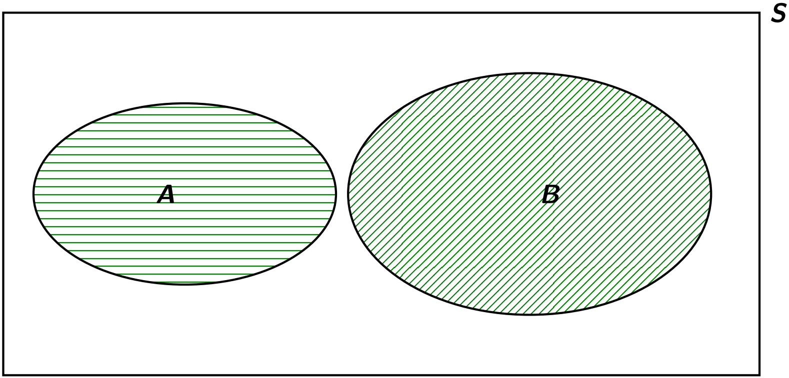
Example 3: Bidding for projects (cont.)

(i) Company *a* submitting a bid for one project and Company *b* submitting a bid for another project:



Example 3: Bidding for projects (cont.)

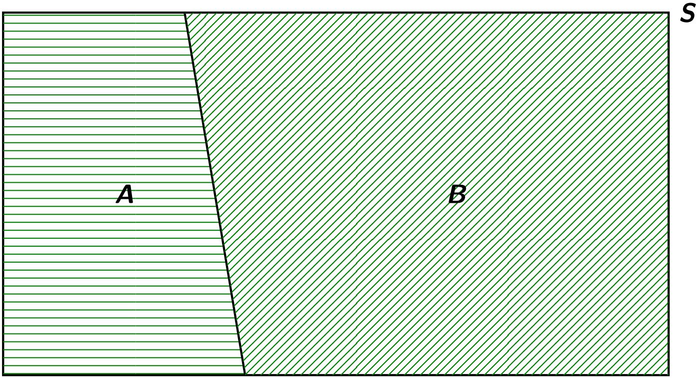
(ii) Companies *a* and *b* submitting bids for the same project.



These events are mutually exclusive, as both companies cannot win the same project.

Example 3: Bidding for projects (cont.)

(iii) Company *a* and company *b* are the only bidders for the single project available.



These events are both mutually exclusive and collectively exhaustive.

Recap

- Set theory and operations: union, intersection, complement
- Events occur within a sample space of possibilities
- The sample space can be discrete or continuous
- Mutually exclusive events cannot jointly occur
- The union of collectively exhaustive events yields the sample space
- De Morgan's Rules are useful for expressing complements of unions or of intersections

Play around with set operations: https://seeing-theory.brown.edu/compound-probability/index.html#section1

De Morgan's rule

Complement of a union

The complement of the union of a given number of sets/events is the intersection of their complements:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad (19)$$

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C} \quad (20)$$

$$\overline{E_1 \cup E_2 \cup \cdots \cup E_n} = \overline{E_1} \cap \overline{E_2} \cap \cdots \cap \overline{E_n} \quad (21)$$

Equivalently:

Complement of an intersection

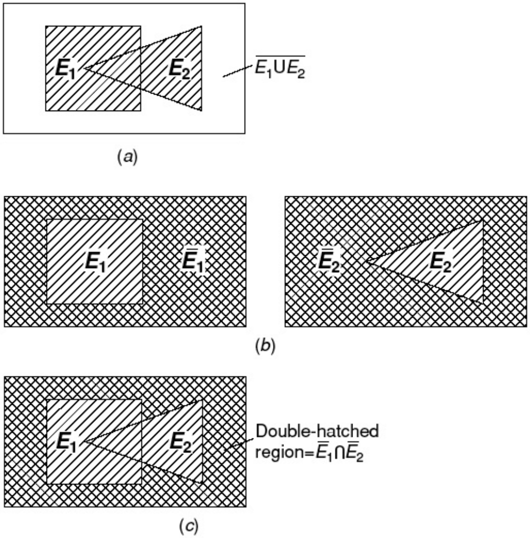
The complement of the intersection of a given number of sets/events is the union of their complements:

$$\overline{A \cap B} = \overline{AB} = \overline{A} \cup \overline{B} \quad (22)$$

$$\overline{A \cap B \cap C} = \overline{ABC} = \overline{A} \cup \overline{B} \cup \overline{C} \quad (23)$$

$$\overline{E_1 \cap E_2 \cap \cdots \cap E_n} = \overline{E_1} \cup \overline{E_2} \cup \cdots \cup \overline{E_n} \quad (24)$$

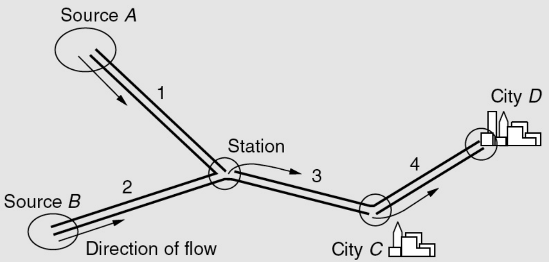
Venn diagram demonstrating de Morgan's rule



$$\overline{E_1 \cup E_2} = \overline{E_1} \cap \overline{E_2}$$

Example 4: Water supply

The water supply for two cities C and D comes from the two sources A and B. Water is transported by pipelines 1, 2, 3 and 4. Assume that either one of the two sources by itself is sufficient to supply the water for both cities. Also, denote E_1, E_2, E_3, E_4 as the failure of branches 1, 2, 3 and 4, respectively.

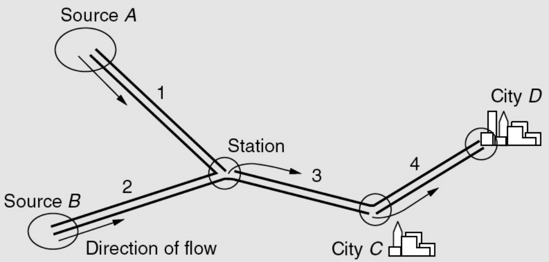


(a) Denote the event that there is no shortage of water in C.

(b) Denote the event that there is no shortage of water in D.

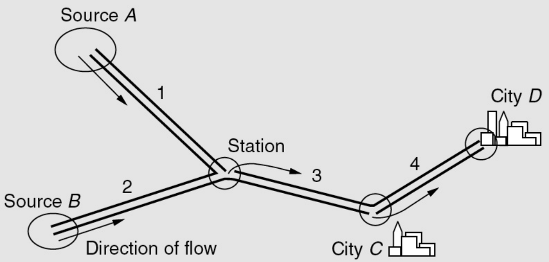
Simplify your answers using De Morgan's rule.

Example 4: Water supply (cont.)



Shortage of water in C is represented by $E_1 \cap E_2 \cup E_3$. Its complement $\overline{E_1 E_2 \cup E_3}$ means there is no shortage of water in C. Applying de Morgan's rule, we have:
$\overline{E_1 E_2 \cup E_3} = \overline{E_1 E_2} \cap \overline{E_3} = (\overline{E_1} \cup \overline{E_2})\overline{E_3}$

Example 4: Water supply (cont.)



No shortage of water in D is represented by $\overline{E_1 E_2 \cup E_3 \cup E_4}$.

Simplified using de Morgan's rule, this becomes $(\overline{E_1} \cup \overline{E_2})\overline{E_3}\,\overline{E_4}$